

POSITIVE ENERGY THEOREM FOR $(4+1)$ -DIMENSIONAL ASYMPTOTICALLY ANTI-DE SITTER SPACETIMES

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ABSTRACT. We define the total energy-momenta for $(4+1)$ -dimensional asymptotically anti-de Sitter spacetimes, and prove the positive energy theorem for such spacetimes.

1. INTRODUCTION

Positive mass theorem is an important result in general relativity. When the cosmological constant is zero and the spacetime is asymptotically flat, the definition of total energy and total linear momentum was first given by Arnowitt, Deser and Misner [1]. Physicists then conjectured that the total mass of an isolated gravity system should be nonnegative. This conjecture was first proved by Schoen and Yau [11, 12, 13] and then by Witten [16]. For the wide applications in geometry and physics, higher dimensional positive mass theorems for asymptotically flat manifolds were extensively studied [3, 18, 7, 8].

When the cosmological constant is negative and the initial data is asymptotically hyperbolic, the corresponding positive mass theorems were studied by many authors. When the second fundamental form is zero, such positive mass theorems were obtained under different conditions [4, 6, 14]. When the second fundamental form is nonzero, such positive energy theorems were given in [9, 17, 15]. For higher dimensional asymptotically anti-de Sitter spacetimes, Chruściel, Maerten and Tod [5] gave a definition for the total energy and other conserved quantities. Under certain assumptions on coordinate transformations, they obtained some inequalities of the total energy. The first author, Xie and Zhang [15] obtained more general inequalities of the total energy, without the assumptions on coordinate transformations.

In this paper, we establish similar inequalities involving total energy-momenta for $(4+1)$ -dimensional asymptotically anti-de Sitter spacetimes. We give the explicit form of the imaginary Killing spinors on the 0-slice of the anti-de Sitter spacetime after fixing a suitable Clifford representation. We define the total energy and momenta for asymptotically anti-de Sitter initial data, and finally, we provide the lower bound of the total energy in terms of the total momenta, which implies the positive energy theorem.

This paper is organized as follows: In Section 2, we give the explicit form of imaginary Killing spinors on the 0-slice of the anti-de Sitter spacetime.

In Section 3, we give the definition of total energy and total momenta. In Section 4, we prove our positive energy theorem. In Appendix, we provide the explicit form of the Killing vectors of the anti-de Sitter spacetime.

2. THE ANTI-DE SITTER SPACE-TIME

The anti-de Sitter (*AdS*) spacetime with negative cosmological constant Λ , denoted by (N, \tilde{g}_{AdS}) , is a static spherically symmetric solution of the vacuum Einstein equations. The $(4+1)$ -dimensional anti-de Sitter spacetime is indeed the hyperboloid

$$\eta_{\alpha\beta} y^\alpha y^\beta = \frac{6}{\Lambda}, \quad \Lambda = -6\kappa^2 (\kappa > 0)$$

in $\mathbb{R}^{4,2}$ with the metric

$$ds^2 = -(dy^0)^2 + \sum_{i=1}^4 (dy^i)^2 - (dy^5)^2. \quad (2.1)$$

Under suitable choice of coordinates, the metric of $(4+1)$ -dimensional anti-de Sitter spacetime can be written as

$$ds^2 = -\cosh^2(\kappa r) dt^2 + dr^2 + \frac{\sinh^2(\kappa r)}{\kappa^2} \left(d\theta^2 + \sin^2 \theta (d\psi^2 + \sin^2 \psi d\varphi^2) \right).$$

The t -slice $(\mathbb{H}^4, \check{g})$ is the hyperbolic 4-space with constant sectional curvature $-\kappa^2$.

Let the orthonormal frame of the AdS spacetime be

$$\begin{aligned} \check{e}_0 &= \frac{1}{\cosh(\kappa r)} \frac{\partial}{\partial t}, \quad \check{e}_1 = \frac{\partial}{\partial r}, \quad \check{e}_2 = \frac{\kappa}{\sinh(\kappa r)} \frac{\partial}{\partial \theta}, \\ \check{e}_3 &= \frac{\kappa}{\sinh(\kappa r) \sin \theta} \frac{\partial}{\partial \psi}, \quad \check{e}_4 = \frac{\kappa}{\sinh(\kappa r) \sin \theta \sin \psi} \frac{\partial}{\partial \varphi}, \end{aligned}$$

and \check{e}^α be the dual coframe of \check{e}_α .

Recall that the fifteen Killing vectors

$$U_{\alpha\beta} = y_\alpha \frac{\partial}{\partial y^\beta} - y_\beta \frac{\partial}{\partial y^\alpha}$$

generate rotations for $\mathbb{R}^{4,2}$ with the metric (2.1). By restricting these vectors to the hyperboloid $\{\eta_{\alpha\beta} y^\alpha y^\beta = \frac{6}{\Lambda}\}$ with the induced metric, the Killing vectors of *AdS* spacetime can be derived. See Appendix for the explicit form of $U_{\alpha\beta}$ on the 0-slice.

Let \mathbb{S} be the spinor bundle of (N, \tilde{g}_{AdS}) and its restriction to \mathbb{H}^4 . The spinor $\Phi_0 \in \Gamma(\mathbb{S})$ is called an imaginary Killing spinor along \mathbb{H}^4 if it satisfies

$$\nabla_X^{AdS} \Phi_0 + \frac{\kappa\sqrt{-1}}{2} X \cdot \Phi_0 = 0$$

for each X tangent to \mathbb{H}^4 .

For the following application, we fix the following Clifford representation throughout this paper:

$$\begin{aligned}\check{e}_0 &\mapsto \begin{pmatrix} I & \\ & -I \end{pmatrix}, \quad \check{e}_1 \mapsto \begin{pmatrix} & I \\ -I & \end{pmatrix}, \quad \check{e}_2 \mapsto \begin{pmatrix} & i \\ i & \end{pmatrix}, \\ \check{e}_3 &\mapsto \begin{pmatrix} & j \\ j & \end{pmatrix}, \quad \check{e}_4 \mapsto \begin{pmatrix} & k \\ k & \end{pmatrix},\end{aligned}\tag{2.2}$$

where

$$i = \begin{pmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{pmatrix}, \quad j = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad k = \begin{pmatrix} & \sqrt{-1} \\ \sqrt{-1} & \end{pmatrix}.$$

Under this representation, we have

Lemma 2.1. *The imaginary Killing spinors along \mathbb{H}^4 are all of the form*

$$\Phi_0 = \begin{pmatrix} u^+ e^{\frac{\kappa r}{2}} + u^- e^{-\frac{\kappa r}{2}} \\ v^+ e^{\frac{\kappa r}{2}} + v^- e^{-\frac{\kappa r}{2}} \\ \sqrt{-1} u^+ e^{\frac{\kappa r}{2}} - \sqrt{-1} u^- e^{-\frac{\kappa r}{2}} \\ \sqrt{-1} v^+ e^{\frac{\kappa r}{2}} - \sqrt{-1} v^- e^{-\frac{\kappa r}{2}} \end{pmatrix},\tag{2.3}$$

where

$$\begin{aligned}u^+ &= \left(\lambda_1 e^{\frac{-\sqrt{-1}}{2}\varphi} \cos \frac{\psi}{2} + \lambda_2 e^{\frac{\sqrt{-1}}{2}\varphi} \sin \frac{\psi}{2} \right) \cos \frac{\theta}{2} \\ &\quad + \left(\lambda_3 e^{\frac{-\sqrt{-1}}{2}\varphi} \cos \frac{\psi}{2} + \lambda_4 e^{\frac{\sqrt{-1}}{2}\varphi} \sin \frac{\psi}{2} \right) \sin \frac{\theta}{2}, \\ u^- &= -\sqrt{-1} \left(\lambda_1 e^{\frac{-\sqrt{-1}}{2}\varphi} \cos \frac{\psi}{2} + \lambda_2 e^{\frac{\sqrt{-1}}{2}\varphi} \sin \frac{\psi}{2} \right) \sin \frac{\theta}{2} \\ &\quad + \sqrt{-1} \left(\lambda_3 e^{\frac{-\sqrt{-1}}{2}\varphi} \cos \frac{\psi}{2} + \lambda_4 e^{\frac{\sqrt{-1}}{2}\varphi} \sin \frac{\psi}{2} \right) \cos \frac{\theta}{2}, \\ v^+ &= \sqrt{-1} \left(-\lambda_1 e^{\frac{-\sqrt{-1}}{2}\varphi} \sin \frac{\psi}{2} + \lambda_2 e^{\frac{\sqrt{-1}}{2}\varphi} \cos \frac{\psi}{2} \right) \cos \frac{\theta}{2} \\ &\quad + \sqrt{-1} \left(\lambda_3 e^{\frac{-\sqrt{-1}}{2}\varphi} \sin \frac{\psi}{2} - \lambda_4 e^{\frac{\sqrt{-1}}{2}\varphi} \cos \frac{\psi}{2} \right) \sin \frac{\theta}{2}, \\ v^- &= \left(\lambda_1 e^{\frac{-\sqrt{-1}}{2}\varphi} \sin \frac{\psi}{2} - \lambda_2 e^{\frac{\sqrt{-1}}{2}\varphi} \cos \frac{\psi}{2} \right) \sin \frac{\theta}{2} \\ &\quad + \left(\lambda_3 e^{\frac{-\sqrt{-1}}{2}\varphi} \sin \frac{\psi}{2} - \lambda_4 e^{\frac{\sqrt{-1}}{2}\varphi} \cos \frac{\psi}{2} \right) \cos \frac{\theta}{2}.\end{aligned}$$

Here $\lambda_1, \lambda_2, \lambda_3$, and λ_4 are arbitrary complex numbers.

3. DEFINITIONS

Suppose that N is a spacetime with the metric \tilde{g} of signature $(-1, 1, 1, 1, 1)$, satisfying the Einstein field equations

$$\widetilde{Ric} - \frac{\tilde{R}}{2}\tilde{g} + \Lambda\tilde{g} = T,\tag{3.1}$$

where \widetilde{Ric} , \widetilde{R} are the Ricci and scalar curvatures of \widetilde{g} respectively, T is the energy-momentum tensor of matter, and Λ is the cosmological constant. For orthonormal frame $\{e_\alpha\}$ with e_0 timelike, the dominant energy condition

$$T_{00} \geq \sqrt{\sum_i T_{0i}^2}, \quad T_{00} \geq |T_{\alpha\beta}| \quad (3.2)$$

is satisfied. Let M be a 4-dimensional spacelike hypersurface in N with the induced metric g and p be the second fundamental form of M in N .

Definition 3.1. *An initial data set (M, g, p) is asymptotically AdS of order $\tau > 2$ if*

- (1) *there is a compact set K such that $M_\infty = M \setminus K$ is diffeomorphic to $\mathbb{R}^4 - B_r$, where B_r is the closed ball of radius r with center at the coordinate origin;*
- (2) *Under the diffeomorphism, $g_{ij} = g(\check{e}_i, \check{e}_j) = \delta_{ij} + a_{ij}$, $h_{ij} = h(\check{e}_i, \check{e}_j)$ satisfy*

$$\begin{aligned} a_{ij} &= O(e^{-\tau\kappa r}), \quad \check{\nabla}_k a_{ij} = O(e^{-\tau\kappa r}), \quad \check{\nabla}_l \check{\nabla}_k a_{ij} = O(e^{-\tau\kappa r}), \\ h_{ij} &= O(e^{-\tau\kappa r}), \quad \check{\nabla}_k h_{ij} = O(e^{-\tau\kappa r}), \end{aligned} \quad (3.3)$$

where $\check{\nabla}$ is the Levi-Civita connection with respect to the hyperbolic metric \check{g} ;

- (3) *there is a distance function ρ_z such that $T_{00}e^{\kappa\rho_z}, T_{0i}e^{\kappa\rho_z} \in L^1(M)$.*

Remark 3.1. *For simplicity, we assume the manifold M has only one end. The results we obtain in the paper could be extended to multi-end case easily.*

Denote

$$\begin{aligned} \mathcal{E}_i &= \check{\nabla}^j g_{ij} - \check{\nabla}_i \text{tr}_{\check{g}}(g) - \kappa(a_{1i} - g_{1i} \text{tr}_{\check{g}}(a)), \\ \mathcal{P}_{ki} &= p_{ki} - g_{ki} \text{tr}_{\check{g}}(p). \end{aligned}$$

Then we can define the following quantities for asymptotically AdS initial data.

Definition 3.2. *For asymptotically AdS initial data, the total energy is defined as*

$$E_0 = \frac{\kappa}{16\pi} \lim_{r \rightarrow \infty} \int_{S_r} \mathcal{E}_1 U_{50}^{(0)} \check{\omega}.$$

The total momenta are defined as

$$\begin{aligned} c_i &= \frac{\kappa}{16\pi} \lim_{r \rightarrow \infty} \int_{S_r} \mathcal{E}_1 U_{i5}^{(0)} \check{\omega}, \\ c'_i &= \frac{\kappa}{8\pi} \sum_{j=2}^4 \lim_{r \rightarrow \infty} \int_{S_r} \mathcal{P}_{j1} U_{i0}^{(j)} \check{\omega}, \\ J_{ij} &= \frac{\kappa}{8\pi} \sum_{j=2}^4 \lim_{r \rightarrow \infty} \int_{S_r} \mathcal{P}_{k1} U_{ij}^{(k)} \check{\omega}, \end{aligned}$$

where

$$\check{\omega} = \check{e}^2 \wedge \check{e}^3 \wedge \check{e}^4, \quad U_{\alpha\beta} = U_{\alpha\beta}^{(\gamma)} \check{e}_\gamma.$$

Remark 3.2. If $\kappa = 1$, similar to [15], we can derive the following relations between the quantities in [5] and the quantities defined in Definition 3.2:

$$\begin{aligned} H(V_{(0)}, 0) &= m_{(0)} = E_0, & H(V_{(i)}, 0) &= m_{(i)} = -c_i, \\ H(0, C_{(i)}) &= c_{(i)} = c'_i, & H(0, \Omega_{(i)(j)}) &= J_{(i)(j)} = J_{ij}, \end{aligned}$$

where $i, j = 1, 2, 3, 4$.

4. POSITIVE ENERGY THEOREM

Suppose (N, \tilde{g}) is a $(4 + 1)$ -dimensional spacetime, and M is an asymptotically AdS hypersurface in N with the induced metric g and the second fundamental form p . $\tilde{\nabla}$ and ∇ are the Levi-Civita connections corresponding to \tilde{g} and g respectively. For simplicity, we also use the same symbols to denote their lifts to the spinor bundle \mathbb{S} respectively. Define

$$\hat{\nabla}_i = \tilde{\nabla}_i + \frac{\sqrt{-1}}{2} \kappa e_i \cdot, \quad \hat{D} = \sum_{i=1}^4 e_i \cdot \hat{\nabla}_i, \quad (4.1)$$

then we can derive the following Weitzenböck formula [17]

$$\hat{D}^* \hat{D} = \hat{\nabla}^* \hat{\nabla} + \hat{\mathcal{R}},$$

with

$$\hat{\mathcal{R}} = \frac{1}{2} (T_{00} e_0 + T_{0i} e_i) \cdot e_0 \cdot.$$

By Lax-Milgram Theorem, it is easy to prove that there exists a unique solution to the equation $\hat{D}\phi = 0$ on M , with ϕ asymptotical to the imaginary Killing spinor Φ_0 on the end [17]. By integrating the Weitzenböck formula (4.1) and applying Witten's argument [10, 2, 19, 17], we get

$$\begin{aligned} & \int_M |\hat{\nabla}\phi|^2 * 1 + \int_M \langle \phi, \hat{\mathcal{R}}\phi \rangle * 1 \\ &= \lim_{r \rightarrow \infty} \operatorname{Re} \int_{S_r} \langle \phi, \sum_{j \neq i} e_i \cdot e_j \cdot \hat{\nabla}_j \phi \rangle * e^i \\ &= \frac{1}{4} \lim_{r \rightarrow \infty} \int_{S_r} (\check{\nabla}^j g_{1j} - \check{\nabla}_1 \operatorname{tr}_{\check{g}}(g)) |\Phi_0|^2 \check{\omega} \\ & \quad + \frac{1}{4} \lim_{r \rightarrow \infty} \int_{S_r} \kappa(a_{k1} - g_{k1} \operatorname{tr}_{\check{g}}(a)) \langle \Phi_0, \sqrt{-1} \check{e}_k \cdot \Phi_0 \rangle \check{\omega} \\ & \quad - \frac{1}{2} \lim_{r \rightarrow \infty} \int_{S_r} (h_{k1} - g_{k1} \operatorname{tr}_{\check{g}}(h)) \langle \Phi_0, \check{e}_0 \cdot \check{e}_k \cdot \Phi_0 \rangle \check{\omega}, \end{aligned} \quad (4.2)$$

where ϕ is the unique solution of the equation $\hat{D}\phi = 0$.

By the Clifford representation (2.2) and the explicit form (2.3) of Φ_0 , the boundary term on the right hand side of (4.2) is equal to

$$\begin{aligned}
RHS &= \frac{1}{2} \lim_{r \rightarrow \infty} \int_{S_r} \mathcal{E}_1 \left(\overline{u^+} u^+ + \overline{v^+} v^+ \right) e^{\kappa r} \check{\omega} \\
&\quad + \lim_{r \rightarrow \infty} \int_{S_r} \mathcal{P}_{21} \left(\overline{u^+} u^+ - \overline{v^+} v^+ \right) e^{\kappa r} \check{\omega} \\
&\quad - \sqrt{-1} \lim_{r \rightarrow \infty} \int_{S_r} \mathcal{P}_{31} \left(\overline{u^+} v^+ - \overline{v^+} u^+ \right) e^{\kappa r} \check{\omega} \\
&\quad + \lim_{r \rightarrow \infty} \int_{S_r} \mathcal{P}_{41} \left(\overline{u^+} v^+ + \overline{v^+} u^+ \right) e^{\kappa r} \check{\omega} \\
&= 8\pi(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4) Q(\lambda_1, \lambda_2, \lambda_3, \lambda_4)^t,
\end{aligned} \tag{4.3}$$

in which the matrix

$$Q = \begin{pmatrix} E & L \\ \bar{L}^t & \hat{E} \end{pmatrix},$$

where

$$\begin{aligned}
E &= \begin{pmatrix} E_0 + c_4 & c'_1 + \sqrt{-1}c'_2 \\ +c'_3 - J_{34} & -J_{14} - \sqrt{-1}J_{24} \\ c'_1 - \sqrt{-1}c'_2 & E_0 + c_4 \\ -J_{14} + \sqrt{-1}J_{24} & -c'_3 + J_{34} \end{pmatrix}, \\
\hat{E} &= \begin{pmatrix} E_0 - c_4 & -c'_1 - \sqrt{-1}c'_2 \\ -c'_3 - J_{34} & -J_{14} - \sqrt{-1}J_{24} \\ -c'_1 + \sqrt{-1}c'_2 & E_0 - c_4 \\ -J_{14} + \sqrt{-1}J_{24} & +c'_3 + J_{34} \end{pmatrix}, \\
L &= \begin{pmatrix} c_3 & c_1 + \sqrt{-1}c_2 \\ -c'_4 + \sqrt{-1}J_{12} & +J_{13} + \sqrt{-1}J_{23} \\ c_1 - \sqrt{-1}c_2 & -c_3 \\ -J_{13} + \sqrt{-1}J_{23} & -c'_4 - \sqrt{-1}J_{12} \end{pmatrix}.
\end{aligned}$$

Set $\hat{J}_k = \frac{1}{2}\varepsilon_{ijk}J_{jk}$ and denote

$$\begin{aligned}
\mathbf{c} &= (c_1, c_2, c_3), \quad \mathbf{c}' = (c'_1, c'_2, c'_3), \quad \hat{\mathbf{J}} = (\hat{J}_1, \hat{J}_2, \hat{J}_3), \quad \mathbf{J}_{(4)} = (J_{14}, J_{24}, J_{34}), \\
|L|^2 &= 2(|\mathbf{c}|^2 + |\hat{\mathbf{J}}|^2 + c_4'^2), \quad A = c_4^2 + c_4'^2 + |\mathbf{c}|^2 + |\mathbf{c}'|^2 + |\hat{\mathbf{J}}|^2 + |\mathbf{J}_{(4)}|^2,
\end{aligned}$$

then we have

Theorem 4.1. *Let (M, g, h) be a 4-dimensional asymptotically anti-de Sitter initial data of the spacetime (N, \tilde{g}) satisfying the dominant energy condition (3.2). Then we have the following inequality:*

$$E_0 \geq \max \left\{ \begin{aligned} & \left(c_4^2 + \frac{1}{4}|L|^2 \right)^{\frac{1}{2}}, \left(\frac{1}{2}(|\mathbf{c}|^2 + |\mathbf{J}_{(4)}|^2) + \frac{1}{8}|L|^2 \right)^{\frac{1}{2}}, (A + |\mathbf{c}'|^2 + |\mathbf{J}_{(4)}|^2)^{\frac{1}{2}} \\ & - |\mathbf{c}'| - |\mathbf{J}_{(4)}|, \left(A - 2\sqrt{2} \left(\sum_{i=1}^3 (c_4 c'_i - c'_4 c_i)^2 + |\mathbf{c} \times \hat{\mathbf{J}}|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}, \\ & \left(A - 4\sqrt{2} \left(\sum_i (c_4 c'_i - c'_4 c_i)^2 + |\mathbf{c} \times \hat{\mathbf{J}}|^2 \right)^{\frac{1}{2}} + F_+^{\frac{1}{2}} \right)^{\frac{1}{2}} \end{aligned} \right\},$$

where

$$F_+ = \max\{F, 0\},$$

$$\begin{aligned} F = & -8\sqrt{2} \left(\sum_{i=1}^3 (c_4 c'_i - c'_4 c_i)^2 + |\mathbf{c} \times \hat{\mathbf{J}}|^2 \right)^{\frac{1}{2}} A + 36|\mathbf{c} \times \hat{\mathbf{J}}|^2 + 4|\mathbf{c} \times \mathbf{c}'|^2 \\ & + 36 \sum_{i=1}^3 (c_4 c'_i - c'_4 c_i)^2 + 4(|\mathbf{J}_{(4)} \cdot \mathbf{c}'|^2 + |\mathbf{J}_{(4)} \cdot \hat{\mathbf{J}}|^2 + |\mathbf{J}_{(4)} \cdot \mathbf{c}|^2) \\ & + 4|\mathbf{c}' \times \hat{\mathbf{J}}|^2 + 4|\mathbf{J}_{(4)}|^2 (c_4^2 + c_4'^2) + 8c_4 \varepsilon_{ijk} c_i \hat{J}_j J_{k4} + 8c_4' \varepsilon_{ijk} c'_i \hat{J}_j J_{k4}. \end{aligned}$$

Moreover, if $E_0 = 0$, then $Q = 0$ and the spacetime (N, \tilde{g}) is anti-de Sitter along M .

Proof: The nonnegativity of the Hermitian matrix Q can be derived from the integral form of the Weitzenböck formula (4.2), (4.3) and the dominant energy condition (3.2).

The nonnegativity of first-order principal minors ensures $E_0 \geq 0$. And from the nonnegativity of second-order principal minors, one finds

$$E_0 \geq \left(c_4^2 + \frac{1}{2} \sum_{i=1}^3 (c_i^2 + \hat{J}_i^2) + \frac{1}{2} c_4'^2 \right)^{\frac{1}{2}} \quad (4.4)$$

and

$$E_0 \geq \left(\frac{1}{2} \sum_{i=1}^3 (c_i'^2 + J_{i4}^2) + \frac{1}{4} \sum_{i=1}^3 (c_i^2 + \hat{J}_i^2) + \frac{1}{4} c_4'^2 \right)^{\frac{1}{2}}. \quad (4.5)$$

The sum of third-order principal minors is given, up to a positive constant, by

$$S = E_0(E_0^2 - A) + 2c_4' \sum_{i=1}^3 c_i J_{i4} + 2\varepsilon_{ijk} c_i c'_j \hat{J}_k - 2c_4 \sum_{i=1}^3 c'_i J_{i4}.$$

Using the Cauchy inequality, one derives

$$S \leq E_0(E_0^2 - A) + 2|c_4'| |\mathbf{c}| |\mathbf{J}_{(4)}| + 2|\mathbf{c}'| |\mathbf{c} \times \hat{\mathbf{J}}| + 2E_0 |\mathbf{c}'| |\mathbf{J}_{(4)}|.$$

Since $|c'_4||\mathbf{c}| \leq \frac{1}{2}(c_4'^2 + |\mathbf{c}|^2) \leq E_0^2$ and $|\mathbf{c} \times \hat{\mathbf{J}}| \leq \frac{1}{2}(|\mathbf{c}|^2 + |\hat{\mathbf{J}}|^2) \leq E_0^2$, one can obtain

$$E_0((E_0 + |\mathbf{c}'| + |\mathbf{J}_{(4)}|)^2 - A - |\mathbf{c}'|^2 - |\mathbf{J}_{(4)}|^2) \geq 0.$$

When $E_0 > 0$, we get

$$E_0 \geq (A + |\mathbf{c}'|^2 + |\mathbf{J}_{(4)}|^2)^{\frac{1}{2}} - |\mathbf{c}'| - |\mathbf{J}_{(4)}|,$$

as $A + |\mathbf{c}'|^2 + |\mathbf{J}_{(4)}|^2 \geq (|\mathbf{c}'| + |\mathbf{J}_{(4)}|)^2$. When $E_0 = 0$, the inequality (4.4), together with the inequality (4.5), shows that $Q = 0$. In this case, the inequality becomes trivial.

Also, we have

$$\begin{aligned} & 2c'_4 \sum_{i=1}^3 c_i J_{i4} + 2\varepsilon_{ijk} c_i c'_j \hat{J}_k - 2c_4 \sum_{i=1}^3 c'_i J_{i4} \\ & \leq 2(|\mathbf{J}_{(4)}|^2 + |\mathbf{c}'|^2)^{\frac{1}{2}} \left(\sum_{i=1}^3 (c_4 c'_i - c'_4 c_i)^2 + |\mathbf{c} \times \hat{\mathbf{J}}|^2 \right)^{\frac{1}{2}} \\ & \leq 2\sqrt{2}E_0 \left(\sum_{i=1}^3 (c_4 c'_i - c'_4 c_i)^2 + |\mathbf{c} \times \hat{\mathbf{J}}|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore,

$$S \leq E_0(E_0^2 - A) + 2\sqrt{2}E_0 \left(\sum_{i=1}^3 (c_4 c'_i - c'_4 c_i)^2 + |\mathbf{c} \times \hat{\mathbf{J}}|^2 \right)^{\frac{1}{2}}.$$

This implies

$$E_0^2 \geq A - 2\sqrt{2} \left(\sum_{i=1}^3 (c_4 c'_i - c'_4 c_i)^2 + |\mathbf{c} \times \hat{\mathbf{J}}|^2 \right)^{\frac{1}{2}}$$

if $E_0 > 0$. The case for $E_0 = 0$ is considered similarly.

The determinant of the matrix is

$$\begin{aligned} \det Q &= (E_0^2 - A)^2 + 8E_0 \sum_{i=1}^3 (c'_4 c_i J_{i4} - c_4 c'_i J_{i4}) + 8E_0 \varepsilon_{ijk} c_i c'_j \hat{J}_k \\ &\quad - 4|\mathbf{c} \times \mathbf{c}'|^2 - 4|\mathbf{c} \times \hat{\mathbf{J}}|^2 - 4|\mathbf{c}' \times \hat{\mathbf{J}}|^2 - 4|\mathbf{J}_{(4)}|^2 (c_4^2 + c_4'^2) \\ &\quad - 4 \sum_{i=1}^3 (c_4 c'_i - c'_4 c_i)^2 - 4(|\mathbf{J}_{(4)} \cdot \mathbf{c}'|^2 + |\mathbf{J}_{(4)} \cdot \hat{\mathbf{J}}|^2 + |\mathbf{J}_{(4)} \cdot \mathbf{c}|^2) \\ &\quad - 8c_4 \varepsilon_{ijk} c_i \hat{J}_j J_{k4} - 8c'_4 \varepsilon_{ijk} c'_i \hat{J}_j J_{k4}. \end{aligned}$$

Since

$$c'_4 \sum_{i=1}^3 c_i J_{i4} + \varepsilon_{ijk} c_i c'_j \hat{J}_k - c_4 \sum_{i=1}^3 c'_i J_{i4} \leq \sqrt{2}E_0 \left(\sum_{i=1}^3 (c_4 c'_i - c'_4 c_i)^2 + |\mathbf{c} \times \hat{\mathbf{J}}|^2 \right)^{\frac{1}{2}},$$

one obtains

$$\begin{aligned}
 \det Q &\leq (E_0^2 - A)^2 + 8\sqrt{2}E_0^2 \left(\sum_{i=1}^3 (c_4 c'_i - c'_4 c_i)^2 + |\mathbf{c} \times \hat{\mathbf{J}}|^2 \right)^{\frac{1}{2}} \\
 &\quad - 4|\mathbf{c} \times \mathbf{c}'|^2 - 4|\mathbf{c} \times \hat{\mathbf{J}}|^2 - 4|\mathbf{c}' \times \hat{\mathbf{J}}|^2 - 4|\mathbf{J}_{(4)}|^2 (c_4^2 + c_4'^2) \\
 &\quad - 4 \sum_{i=1}^3 (c_4 c'_i - c'_4 c_i)^2 - 4(|\mathbf{J}_{(4)} \cdot \mathbf{c}'|^2 + |\mathbf{J}_{(4)} \cdot \hat{\mathbf{J}}|^2 + |\mathbf{J}_{(4)} \cdot \mathbf{c}|^2) \\
 &\quad - 8c_4 \varepsilon_{ijk} c_i \hat{J}_j J_{k4} - 8c_4' \varepsilon_{ijk} c'_i \hat{J}_j J_{k4} \\
 &= \left(E_0^2 - A + 4\sqrt{2} \left(\sum_{i=1}^3 (c_4 c'_i - c'_4 c_i)^2 + |\mathbf{c} \times \hat{\mathbf{J}}|^2 \right)^{\frac{1}{2}} \right)^2 \\
 &\quad + 8\sqrt{2} \left(\sum_{i=1}^3 (c_4 c'_i - c'_4 c_i)^2 + |\mathbf{c} \times \hat{\mathbf{J}}|^2 \right)^{\frac{1}{2}} A - 36|\mathbf{c} \times \hat{\mathbf{J}}|^2 - 4|\mathbf{c} \times \mathbf{c}'|^2 \\
 &\quad - 36 \sum_i (c_4 c'_i - c'_4 c_i)^2 - 4(|\mathbf{J}_{(4)} \cdot \mathbf{c}'|^2 + |\mathbf{J}_{(4)} \cdot \hat{\mathbf{J}}|^2 + |\mathbf{J}_{(4)} \cdot \mathbf{c}|^2) \\
 &\quad - 4|\mathbf{c}' \times \hat{\mathbf{J}}|^2 - 4|\mathbf{J}_{(4)}|^2 (c_4^2 + c_4'^2) - 8c_4 \varepsilon_{ijk} c_i \hat{J}_j J_{k4} - 8c_4' \varepsilon_{ijk} c'_i \hat{J}_j J_{k4}.
 \end{aligned}$$

This implies

$$E_0^2 \geq A - 4\sqrt{2} \left(\sum_i (c_4 c'_i - c'_4 c_i)^2 + |\mathbf{c} \times \hat{\mathbf{J}}|^2 \right)^{\frac{1}{2}} + F_+^{\frac{1}{2}}.$$

The inequality claimed in the theorem follows immediately.

The rigidity part can be proved by following the argument in [15]. Here we skip the details. Q.E.D.

Remark 4.1. *If $c_i = 0, i = 1, 2, 3, 4$ and $c'_2 = c'_4 = J_{13} = J_{23} = J_{14} = J_{24} = 0$ after suitable coordinate transformation, the inequality for (4 + 1)-dimensional case in Theorem 2 of [5] can be derived from Theorem 4.1.*

5. APPENDIX

$$\begin{aligned}
U_{50} &= \kappa^{-1} \frac{\partial}{\partial t}, \\
U_{10} &= \kappa^{-1} \sin \theta \sin \psi \cos \varphi \frac{\partial}{\partial r} + \coth(\kappa r) \left(\cos \theta \sin \psi \cos \varphi \frac{\partial}{\partial \theta} \right. \\
&\quad \left. + \frac{\cos \psi \cos \varphi}{\sin \theta} \frac{\partial}{\partial \psi} - \frac{\sin \varphi}{\sin \theta \sin \psi} \frac{\partial}{\partial \varphi} \right), \\
U_{20} &= \kappa^{-1} \sin \theta \sin \psi \sin \varphi \frac{\partial}{\partial r} + \coth(\kappa r) \left(\cos \theta \sin \psi \sin \varphi \frac{\partial}{\partial \theta} \right. \\
&\quad \left. + \frac{\cos \psi \sin \varphi}{\sin \theta} \frac{\partial}{\partial \psi} + \frac{\cos \varphi}{\sin \theta \sin \psi} \frac{\partial}{\partial \varphi} \right), \\
U_{30} &= \kappa^{-1} \sin \theta \cos \psi \frac{\partial}{\partial r} + \coth(\kappa r) \left(\cos \theta \cos \psi \frac{\partial}{\partial \theta} - \frac{\sin \psi}{\sin \theta} \frac{\partial}{\partial \psi} \right), \\
U_{40} &= \kappa^{-1} \cos \theta \frac{\partial}{\partial r} - \coth(\kappa r) \sin \theta \frac{\partial}{\partial \theta}, \\
U_{15} &= \kappa^{-1} \tanh(\kappa r) \sin \theta \sin \psi \cos \varphi \frac{\partial}{\partial t}, \\
U_{25} &= \kappa^{-1} \tanh(\kappa r) \sin \theta \sin \psi \sin \varphi \frac{\partial}{\partial t}, \\
U_{35} &= \kappa^{-1} \tanh(\kappa r) \sin \theta \cos \psi \frac{\partial}{\partial t}, \\
U_{45} &= \kappa^{-1} \tanh(\kappa r) \cos \theta \frac{\partial}{\partial t}, \\
U_{12} &= \frac{\partial}{\partial \varphi}, \\
U_{13} &= -\cos \varphi \frac{\partial}{\partial \psi} + \frac{\cos \psi \sin \varphi}{\sin \psi} \frac{\partial}{\partial \varphi}, \\
U_{14} &= -\sin \psi \cos \varphi \frac{\partial}{\partial \theta} - \frac{\cos \theta \cos \psi \cos \varphi}{\sin \theta} \frac{\partial}{\partial \psi} + \frac{\cos \theta \sin \varphi}{\sin \theta \sin \psi} \frac{\partial}{\partial \varphi}, \\
U_{23} &= -\sin \varphi \frac{\partial}{\partial \psi} - \frac{\cos \psi \cos \varphi}{\sin \psi} \frac{\partial}{\partial \varphi}, \\
U_{24} &= -\sin \psi \sin \varphi \frac{\partial}{\partial \theta} - \frac{\cos \theta \cos \psi \sin \varphi}{\sin \theta} \frac{\partial}{\partial \psi} - \frac{\cos \theta \cos \varphi}{\sin \theta \sin \psi} \frac{\partial}{\partial \varphi}, \\
U_{34} &= -\cos \psi \frac{\partial}{\partial \theta} + \frac{\cos \theta \sin \psi}{\sin \theta} \frac{\partial}{\partial \psi}.
\end{aligned}$$

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